Illustration of Spreading Out

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I am writing out the following standard results to illustrate some basic principles of descent, spreading out, and specialization. The general source of the following techniques is EGA IV_3, section 8.

1 Geometric properties

Throughout this section, X will be a scheme over a field k.

Theorem 1. *Suppose that for any finite separable extension k'/k, the k'-scheme $X_{k'}$ is connected. Then for any field extension K/k, the K-scheme X_K is connected.*

Proof. Let K/k be any field extension, and let $A = \Gamma(X, \mathcal{O}_X)$. For X_K to be connected, it is necessary and sufficient that $\Gamma(X_K, \mathcal{O}_{X_K})$ contains no nontrivial idempotents. Recall that any module over a field is flat, so by flat base change for quasicoherent sheaves we have

$$\Gamma(X_K, \mathcal{O}_{X_K}) = A \otimes_k K.$$

Write $K = \varinjlim K_\alpha$ for K_α finitely generated subextensions of K/k. Since direct limit commutes with tensor product, we have

$$\Gamma(X_K, \mathcal{O}_{X_K}) = \varinjlim A \otimes_k K_\alpha.$$

If $R = \varinjlim R_\beta$, then it is straightforward to check $\mathrm{Idemp}(R) = \varinjlim \mathrm{Idemp}(R_\beta)$, where $\mathrm{Idemp}(R)$ is the set of idempotent elements in the ring R. So to check that $A \otimes_k K$ has no nontrivial idempotents, it suffices to check this for the rings $A \otimes_k K_\alpha$, and we may assume that K/k is a finitely generated extension.

Now since K/k is a finitely generated field extension, there is some integral domain B, finite type over k, such that K is the fraction field of B. As before, for any $f \in B$ we have

$$\Gamma(X_{B_f}, \mathcal{O}_{X_{B_f}}) = A \otimes_k B_f.$$

Since $K = \varinjlim B_f$, the limit being taken over all $f \in B \setminus \{0\}$, the same reasoning as before shows that it suffices to show that $A \otimes_k B$ contains no nontrivial idempotents for all domains B finite type over k.

Suppose that $e \in A \otimes_k B$ is a nonzero idempotent. By the Nullstellensatz, $\bigcap_{\mathfrak{m} \subset B \text{ maximal}} \mathfrak{m} = 0$, so that B injects into the product of its residue fields at maximal ideals. Moreover, the kernel of the map $A \otimes_k B \to \prod_{\mathfrak{m}} A \otimes_k B/\mathfrak{m}$ is equal to $\bigcap_{\mathfrak{m}} A \otimes_k \mathfrak{m} = A \otimes_k \bigcap_{\mathfrak{m}} \mathfrak{m} = 0$. Hence to check $e = 1$, it suffices to check this after quotienting $A \otimes_k B$ by an ideal of the form $A \otimes_k \mathfrak{m}$, for $\mathfrak{m} \subset B$ a maximal ideal. By (another form of) the Nullstellensatz, $B/\mathfrak{m}$ is a finite extension of k for each such $\mathfrak{m}$, so we have reduced to the case that $B = \ell$ is a finite extension of k.

Decompose the extension ℓ/k into $\ell/\ell_{\text{sep}}/k$, where ℓ_{sep}/k is a finite separable extension and ℓ/ℓ_{sep} is purely inseparable. By assumption, $A \otimes_k \ell_{\text{sep}}$ contains no nontrivial idempotents, so we may reduce to the case that ℓ/k is a finite purely inseparable extension. For k of characteristic zero we are now done, so assume $\text{char}(k) = p > 0$. Filtering the extension ℓ/k into a sequence of degree p extensions, we may assume $[\ell : k] = p$, generated by some element $x \in \ell$ with $x^p = c \in k$.

Finally suppose that $e \in A \otimes_k \ell$ is an idempotent, and write $e = \sum_{i=0}^{p-1} a_i \otimes x^i$. By assumption $e^2 = e$, so that $e^p = e$ by induction and hence

$$e = e^p = \sum_{i=0}^{p-1} a_i^p \otimes x^{ip} = \sum_{i=0}^{p-1} a_i^p \otimes c^i = \left(\sum_{i=0}^{p-1} a_i^p c^i \right) \otimes 1.$$

So in fact $e \in A$. By hypothesis A contains no nontrivial idempotents, so we are done! $\square$

This result is by no means optimal; here one can replace K/k by any connected k-scheme, for instance. (It is a reasonable exercise to think through how one must change the preceding proof in order to obtain this.) The following result is not optimal either.

Theorem 2. *Suppose that for every finite purely inseparable extension k'/k, the k'-scheme $X_{k'}$ is reduced. Then for any field extension K/k, the K-scheme X_K is reduced.*

Proof. Reducedness may be checked affine-locally, so we may assume that $X = \operatorname{Spec} A$ is affine. Write $A = \varinjlim_i A_i$ as the direct limit of its finitely-generated k-subalgebras. Since $A_i \subset A$, it follows that $(A_i)_K \subset A_K$ for every field extension K/k. This has two major consequences: (1) the hypothesis implies that $(A_i)_{k'}$ is reduced for every finite purely inseparable extension k'/k, and (2) to check that A_K is reduced for every K/k, it suffices to check this for $(A_i)_K$ (since a direct limit of reduced rings is reduced), so we may assume that A is finitely generated over k. For the same reason, we need only check the result for K/k a finitely generated extension.

Write $K = \operatorname{Frac} B$ for an integral domain B of finite type over k. Since any localization of a reduced ring is reduced, it is sufficient to prove that $A \otimes_k B$ is reduced. If $x \in A \otimes_k B$ is a nilpotent element, then to check $x = 0$ it suffices as in the proof of Theorem 1 to show that $x = 0$ in $A \otimes B/\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of B. Now the reduction of x is a nilpotent element of $A \otimes_k B/\mathfrak{m}$, so to prove the result we may assume that $K = \ell/k$ is a finite extension.

Decompose ℓ/k into $\ell/\ell_i/k$, where ℓ_i/k is purely inseparable and ℓ/ℓ_i is separable. By hypothesis, X_{ℓ_i} is reduced, so we may assume that ℓ/k is a finite separable extension. Via the same argument as above, it suffices now to check that $A/\mathfrak{m} \otimes_k \ell$ is reduced for every maximal ideal $\mathfrak{m} \subset A$, so we may assume that $A = k'$ is a finite extension of k.

By the primitive element theorem, we may write $\ell = k[T]/(f(T))$ for some separable polynomial $f(T) \in k[T]$, so that $k' \otimes_k \ell = k'[T]/(f(T))$. Factoring $f(T) = f_1(T) \cdots f_n(T)$ into pairwise distinct irreducible separable polynomials $f_i(T) \in k'[T]$, it follows from the Chinese Remainder Theorem that $k' \otimes_k \ell = \prod_{i=1}^n k'[T]/(f_i(T))$ is a direct product of separable extensions of k', hence is reduced. $\square$

Exercise. Suppose that $X_{k'}$ is irreducible for every finite separable extension k'/k. Then X_K is irreducible for every field extension K/k. (Hint: for a ring R, $\operatorname{Spec} R$ is irreducible if and only if every zero divisor of R is nilpotent.)

Corollary. *Suppose that $X_{k'}$ is an integral scheme for every finite extension k'/k. Then X_K is integral for every extension K/k.*

Proof. By hypothesis, $X_{k'}$ is irreducible for every finite separable extension k'/k and is reduced for every finite purely inseparable such extension, so we may apply Theorem 2 and the Exercise. $\square$

2 Review of constructible sets

We now move to some results of a different flavor than the preceding. For this we will need the notion of constructible and locally constructible sets. We will not prove any of the following facts, as they are fairly technical and likely to be distracting. However, we will give some sample applications to see how they are used. Before this we briefly review the notion of finitely presented morphisms, also without proof. Note that much of the discussion below simplifies in the locally noetherian case; we indicate these simplifications briefly in what follows.

Definition. We say that a morphism of schemes $f : X \to Y$ is **locally of finite presentation** if for every affine open $U = \operatorname{Spec} A \subset Y$ and $x \in f^{-1}(U)$, there is some affine open neighborhood $\operatorname{Spec} B$ of x such that the induced map $A \to B$ is finitely presented, i.e., there is some surjection $A[T_1, \ldots, T_n] \to B$ with finitely generated kernel. In fact, it is equivalent that every surjection of this form has a finitely generated kernel. In the case that Y is locally noetherian, it is equivalent that f be locally of finite type.

The morphism $f : X \to Y$ is **of finite presentation** or **finitely presented** if it is quasicompact, quasiseparated, and locally of finite presentation. Again, if Y is locally noetherian then it is equivalent that f be finite type. The following lemma summarizes the properties of finitely presented morphisms that we will use.

Lemma 1. *Let $f : X \to Y$ be a finitely presented morphism of schemes.*

1. *If $Y' \to Y$ is an arbitrary morphism, then the induced map $f' : X \times_Y Y' \to Y'$ is also of finite presentation.*
2. *If $g : Y \to Z$ is finitely presented, then $g \circ f : X \to Z$ is also finitely presented.*
3. *If $h : X' \to Y$ is finitely presented and $u : X \to X'$ is a Y-morphism, then u is of finite presentation.*

Definition. If X is a scheme, we say that an open set $U \subset X$ is **retrocompact** if, for every quasicompact open $V \subset X$, the intersection $U \cap V$ is quasicompact. Note that the open set U is retrocompact if and only if the inclusion map $U \to X$ is a quasicompact morphism. In particular, if X is locally noetherian then every open subset is retrocompact.

We say that $C \subset X$ is **constructible** if it is a finite union of sets of the form $U \cap (X \setminus V)$, where U and V are retrocompact open subsets of X. It is a simple exercise to check that the collection of constructible subsets of X is closed under finite unions and complements. If X is locally noetherian then the constructible subsets of X are precisely the finite unions of locally closed subsets of X.

Finally, $C \subset X$ is **locally constructible** if for every $x \in X$ there is some open neighborhood U of x such that $C \cap U$ is a constructible subset of U. Again, the collection of locally constructible subsets of X is closed under finite unions and complements. If X is affine, then every locally constructible set is constructible.

The importance of locally constructible sets comes from the following lemma, which we will prove in an appendix.

Lemma 2. *Let X be a scheme, and let $C \subset X$ be a locally constructible subset.*

1. *The set C is open if and only if it is closed under generization, i.e., if $x \in C$ and $y \in X$ is such that $x \in \overline{\{y\}}$, then $y \in C$.*
2. *(Chevalley's theorem) If $f : X \to Y$ is a finitely presented morphism, then $f(C) \subset Y$ is locally constructible.*
3. *If X is quasicompact and quasiseparated and C is constructible, then there is some affine scheme X' and some finitely presented morphism $f : X' \to X$ such that $f(X') = C$.*

Constructible and locally constructible sets are used in a few different ways. Here is a simple application. We will see a similar application later in Lemma 5.

Theorem 3. *(Nullstellensatz) Let k be a field and suppose that K is a finite type k-algebra which is a field. Then K is a finite extension of k.*

Proof. Since K is finite type over k, it suffices to prove that the extension K/k is algebraic. If not, then there is some injective k-algebra map $k[T] \to K$, which is clearly finite type, and hence finitely presented since everything in sight is noetherian. The induced map $\operatorname{Spec} K \to \mathbf{A}^1_k$ has image $\{(0)\}$, which is ensured to be constructible by Chevalley's theorem. But the constructible subsets of $\mathbf{A}^1_k$ are exactly the finite unions of locally closed subsets, so any constructible set containing (0) must contain a dense open. Since $\{(0)\}$ is not open, this is a contradiction and thus K/k is algebraic. $\square$

Another view on locally constructible sets is furnished by part 1 of Lemma 2: in some sense, they are halfway toward being open. Namely, if one wants to prove that a certain set is open, it suffices to prove that it is locally constructible (which can usually be done by cleverly applying Chevalley's theorem) and then to prove that it is closed under generization. For example, this is the strategy used in the proof of the following theorem.

Theorem 4. *Let $f : X \to S$ be a flat map which is locally of finite presentation. Then f is an open map.*

Proof. Openness of a morphism is evidently affine local on the source and target, so we may assume that $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$, where the map $A \to B$ is flat and finitely presented. Similarly, we are reduced to showing that the image of f is open. By Chevalley's theorem on constructibility, the image of f is then constructible. By part 1 of Lemma 2, we reduce to showing that, if $x \in X$ and $f(x) \in \overline{\{s\}}$ for some $x \in S$, then s lies in the image of f. We may localize to consider the map $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{S,f(x)}$ to reduce to the well-known algebraic fact that a flat local homomorphism of local rings is faithfully flat. $\square$

3 Basic limit formalism

To get more results, we will need some finer information on what happens when one descends maps through limits. I will cite the following result without proof but it should no longer seem so surprising. Do not be intimidated by the attribution; it is truly an elementary result, albeit quite tedious to prove. With some work, one can reduce to the affine case because all of the schemes involved can be covered by finitely many affine opens, each pairwise intersection of which can also be covered by finitely many affine opens.

The key idea that allows this reduction to be made appears in the third paragraph of the proof of Lemma 4 below, which will also feature our third essentially distinct proof technique involving constructible sets. (The moral in the proof is "if a limit of constructible sets is empty, then it is empty at some finite stage.") After that there is a relatively simple argument to be made in commutative algebra.

Lemma 3. *(EGA IV, Theorem 8.5.2, Theorem 8.8.2.) Let $\{S_\lambda\}$ be a projective system of quasicompact quasiseparated schemes with affine transition maps, and let $S = \varprojlim S_\lambda$. Let $\mathcal{F}$ be a quasicoherent sheaf of finite presentation on S. Then there exists some λ_0 and a quasicoherent sheaf of finite presentation $\mathcal{F}_{\lambda_0}$ on S_{λ_0} such that $\mathcal{F}$ is isomorphic to the pullback of $\mathcal{F}_{\lambda_0}$ to S. Moreover, if $\mathcal{G}_{\lambda_0}$ is another quasicoherent sheaf on S_{λ_0} and we let $\mathcal{G}_\lambda$ be the pullback of $\mathcal{G}_{\lambda_0}$ to S_λ for all $\lambda > \lambda_0$ and similarly for $\mathcal{G}$, $\mathcal{F}_\lambda$, then the natural map*

$$\varinjlim \operatorname{Hom}_{S_\lambda}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \to \operatorname{Hom}_S(\mathcal{F}, \mathcal{G})$$

is bijective.

Similarly, let X be a finitely presented S-scheme. Then there exists some λ_0 and a finitely presented S_{λ_0}-scheme X_{λ_0} such that $X = X_{\lambda_0} \times_{S_{\lambda_0}} S$. If Y_{λ_0} is another finitely presented S_{λ_0}-scheme and we set $Y_\lambda = Y_{\lambda_0} \times_{S_{\lambda_0}} S_\lambda$ for all $\lambda > \lambda_0$ and $Y = Y_{\lambda_0} \times_{S_{\lambda_0}} S$ and similarly for X_λ (so that, e.g., $Y = \varprojlim_{\lambda > \lambda_0} Y_\lambda$), then the natural map

$$\varinjlim \operatorname{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Hom}_S(X, Y)$$

is bijective.

This lemma may seem abstract, but it has very concrete consequences. A few sorts of inverse systems to keep in mind are the following:

- Let S_0 be a scheme, let $s \in S_0$ be a point, and let $\{S_\lambda\}$ be the projective system of all affine open neighborhoods of s in S_0, where the transition maps are given by inclusions. Then each S_λ, being affine, is quasicompact and quasiseparated. Moreover, morphisms of affine schemes are affine, so all transition maps are affine. Finally, if $S = \operatorname{Spec} \mathcal{O}_{S_0,s}$, then $S = \varprojlim_\lambda S_\lambda$. So the lemma in this case states that various objects over the local ring of a point in a scheme "spread out" to an open neighborhood of the point. For example, if $\mathcal{F}$ is a finitely presented quasicoherent sheaf over $\operatorname{Spec} \mathcal{O}_{S_0,s}$, then there is an open neighborhood U of s in S_0 and a finitely presented quasicoherent sheaf $\tilde{\mathcal{F}}$ over U so that $\mathcal{F}$ is the stalk of $\tilde{\mathcal{F}}$ at s. All maps between such quasicoherent sheaves $\mathcal{F}$ also "spread out" over some open neighborhood of s, and such "spreading outs" are locally unique, i.e., any two agree after restriction to a sufficiently small open neighborhood of s. The statements for schemes over S have similar interpretations.

- Let $S = \operatorname{Spec} A$ be an affine scheme, and write $A = \varinjlim A_i$ for the finitely generated $\mathbf{Z}$-subalgebras A_i of A. Then if $S_i = \operatorname{Spec} A_i$ we have $S = \varprojlim_i S_i$ and each S_i is quasicompact and quasiseparated. So the lemma states roughly that any sufficiently finite object over a ring "comes from" an object over a finitely generated $\mathbf{Z}$-subalgebra. In particular, using base change results this allows many local questions about schemes to be reduced to questions about noetherian schemes. However, even in the noetherian case this often allows reduction to finite type $\mathbf{Z}$-schemes, which have many very special features. We will see very striking applications of this in Theorems 6 and 9.

- Let $S_0 = \operatorname{Spec} k$ for a field k and let $S = \operatorname{Spec} K$ for K/k an arbitrary field extension. Then $K = \varinjlim k_i$ for k_i the subfields of K which are finitely generated over k. The lemma then allows for various questions about large field extensions K/k to be reduced to the finitely generated case. We saw this method used in section 1, where it was combined with the first item to "spread out" constructions over finitely generated field extensions of k to constructions over finitely generated k-algebras.

It is useful to keep these examples in mind when interpreting the following result.

Lemma 4. *Suppose that $A = \varinjlim_{i \in I} A_i$ for rings A and A_i, and suppose that X_{i_0}, Y_{i_0} are finitely presented A_{i_0}-schemes, and $u_{i_0} : X_{i_0} \to Y_{i_0}$ is an A_{i_0}-morphism for some $i_0 \in I$. For each $i > i_0$, let $X_i = X_{i_0} \otimes_{A_{i_0}} A_i$, $Y_i = Y_{i_0} \otimes_{A_{i_0}} A_i$, and $u_i : X_i \to Y_i$ the induced morphism, and similarly for X, Y, and u.*

If u is

1. *radicial,*
2. *surjective,*

3. an isomorphism,

4. a monomorphism,

5. an open immersion,

6. a closed immersion,

7. separated, or

8. proper,

then there is some $i > i_0$ such that u_i is the same.

Proof. This result is most naturally proved in a longer development, in which the same methods are used to prove many related results, but this is all we need in what follows. For more, see [EGA IV.8]. By the characterization of the radicial property mentioned above, we may replace Y_{i_0} and u_{i_0} by $X_{i_0} \times_{Y_{i_0}} X_{i_0}$ and $\Delta_{X_{i_0}/Y_{i_0}}$ to reduce 1 to 2. (Indeed, $X_{i_0} \times_{Y_{i_0}} X_{i_0}$ is finitely presented since the first projection pr_1 is finitely presented by part 1 of Lemma 1 and the composition of two finitely presented morphisms is finitely presented by part 2 of Lemma 1.)

First note that each u_i is finitely presented since X_i and Y_i are, and Chevalley's theorem holds for all finitely presented morphisms. So for each i, the set $C_i = u_i(X_i) \subset Y_i$ is locally constructible, and moreover if $j > i$ then $u_j^{-1}(C_i) = C_j$. For the same reason, $u^{-1}(C_j) = Y$ for all $j > i_0$ because u is surjective. Now let $Z_j = Y_j \setminus C_j$, so that Z_j is locally constructible and $u_j^{-1}(Z_j) = \emptyset$ for all j; the claim is now that $Z_j = \emptyset$ for all large j.

Since each Z_{i_0} is locally constructible in the quasicompact quasiseparated Y_{i_0}, there is some finitely presented affine A_{i_0}-scheme $S_{i_0} = \operatorname{Spec} B_{i_0}$ and some map $f_{i_0} : S_{i_0} \to Y_{i_0}$ such that $Z_{i_0} = f_{i_0}(S_{i_0})$ by part 3 of Lemma 2. For each i we have $S_i = \operatorname{Spec} B_i = \operatorname{Spec} B_{i_0} \otimes_{A_{i_0}} A_i$ and similarly $S = \operatorname{Spec} B = \operatorname{Spec} B_{i_0} \otimes_{A_{i_0}} A$, with similar notation to the preceding. This gives that $f(S) = \emptyset$, so that $S = \emptyset$, i.e., $B = 0$. But since $B = \varinjlim_j (B_{i_0} \otimes_{A_{i_0}} A_j)$ and since a direct limit of rings can only be 0 if it is 0 at some finite stage (pay attention to where 1 is sent!), it follows that indeed $Z_j = f_j(S_j) = \emptyset$ for some large j. This completes the proofs of 1 and 2.

Since u is a monomorphism if and only if $\Delta_{X/Y}$ is an isomorphism, 4 reduces to 3. So suppose that u is an isomorphism and let $v : Y \to X$ be its inverse. By Lemma 3, v descends to some map $v_i : Y_i \to X_i$, and because $u_i \circ v_i$ and $v_i \circ u_i$ are both the respective identity morphisms, it follows from another application of Lemma 3 that there is some $j > i$ such that $u_j \circ v_j = \mathrm{id}_{Y_j}$ and $v_j \circ u_j = \mathrm{id}_{X_j}$. Thus u_j is an isomorphism.

For 5, we prove first that if u is an open immersion then $u_i(X_i) \subset Y_i$ is open for large i. Indeed, for this we may assume that $X_{i_0} = \operatorname{Spec} C_{i_0}$ and $Y_{i_0} = \operatorname{Spec} B_{i_0}$ are affine and then that $u(X) \subset Y$ is a distinguished open, say equal to $\operatorname{Spec} B_f$ for some $f \in B$. Let $i > i_0$ be large enough that f comes from some $f_i \in B_i$, so that $V_i = \operatorname{Spec}(B_i)_{f_i} \subset Y_i$ is an affine open whose pullback to Y is equal to $u(X)$. Then the symmetric difference $u_i(X_i)\Delta V_i$ is locally constructible and its pullback to Y is empty, so by the same argument concerning limits and constructibility as before we may increase i to make $u_i(X_i) = \operatorname{Spec}(B_i)_{f_i}$. In particular, $u_i(X_i)$ is open for large i.

Set $U_i = u_i(X_i)$ and $U = u(X)$, so that after replacing X_i by U_i and X by U we may assume that u is an isomorphism. This reduces us to 3, which we have already proved.

Next suppose that u is a closed immersion. By the same argument as in the open immersion case, $u_i(X_i)$ is closed in Y_i for large i. The argument works much as in the open immersion case, but the problem here is how to put a closed subscheme structure on $u_i(X_i)$ in such a way that its base change to S is equal to $u(X)$. The structure sheaf on $u(X)$ as a closed subscheme of Y is a finitely presented quasicoherent quotient of $\mathcal{O}_Y$, so for large i there exists some finitely presented quasicoherent sheaf $\mathcal{F}_i$ on Y_i such that the pullback of $\mathcal{F}_i$ to Y is isomorphic to $\mathcal{O}_{u(X)}$. Moreover, by increasing i we may assume that there is a map $\varphi_i : \mathcal{O}_{Y_i} \to \mathcal{F}_i$ which pulls back to the natural quotient map. Thus everything depends on showing that one can increase i to make φ_i surjective.

Because we now have a map, the fact that Y_i is quasicompact allows us to pass to an affine open cover of Y_i and thus assume that Y_i is affine. Let $\mathcal{C}_i$ denote the cokernel of φ_i, so that $\mathcal{C}_j$ is the cokernel of φ_j for all $j > i$. The natural map

$$\varinjlim \Gamma(Y_j, \mathcal{C}_j) \to \Gamma(Y, \mathcal{C}) = 0$$

is bijective by Lemma 3 since, e.g., $\Gamma(Y_j, \mathcal{C}_j) = \operatorname{Hom}(\mathcal{O}_{Y_j}, \mathcal{C}_j)$. Since $\Gamma(Y_j, \mathcal{C}_j)$ is a finitely presented $\Gamma(Y_j, \mathcal{O}_{Y_j})$-module, it follows that there is some $j > i$ such that $\Gamma(Y_j, \mathcal{C}_j) = 0$. Since Y_j is affine it follows that $\mathcal{C}_j = 0$, i.e., φ_j is surjective.

The claim for separatedness follows formally from 6 because a morphism is separated if and only if the relative diagonal morphism is a closed immersion. So we are reduced to only the claim for properness. Suppose that u is proper.

By 7, there is some i_0 such that u_{i_0} is separated. So by Chow's Lemma for finitely presented separated morphisms, there exists a commutative diagram

$$\begin{tikzcd} X_{i_0} & Z_{i_0} \arrow[l, "f_{i_0}"'] \arrow[r, "j_{i_0}"] \arrow[d] & P_{i_0} \arrow[dl, "g_{i_0}"] \\ & Y_{i_0} \arrow[from=1-1, "u_{i_0}"'] & \end{tikzcd}$$

where P_{i_0} is a finitely presented A_{i_0}-scheme, j_{i_0} is an open immersion, f_{i_0} is projective and surjective, and g_{i_0} is projective. By 6, there is some $i_1 > i_0$ such that j_{i_1} is a closed immersion, using the obvious notation. In that case $g_{i_1} \circ j_{i_1}$ is projective and in particular proper. It is then easy to see that u_{i_1} is a closed map. Since all properties of this diagram are preserved by arbitrary base change, it follows that u_{i_1} is universally closed. Since u_{i_1} is separated and finite type, it follows that it is proper. □

There is one loose end in the preceding proof, namely the version of Chow's Lemma for finitely presented separated morphisms.

Lemma 5. *Let A be a ring, let X and Y be two finitely presented A-schemes, and let $u : X \to Y$ be a separated morphism. Then there exists a finitely presented A-scheme P and a commutative diagram*

$$\begin{tikzcd} X & Z \arrow[l, "f"'] \arrow[r, "j"] \arrow[d] & P \arrow[dl, "g"] \\ & Y \arrow[from=1-1, "u"'] & \end{tikzcd}$$

where j is an open immersion, f is projective and surjective, and g is projective.

Proof. Since X and Y are finitely presented over A, Lemma 3 and part 7 of Lemma 4 guarantee the existence of a noetherian subring $A' \subset A$, finite type A'-schemes X' and Y', and a separated morphism $u' : X' \to Y'$ which base changes to $u : X \to Y$. By the usual form of Chow's lemma, there exists a finite type A'-scheme P' and a commutative diagram

$$\begin{tikzcd} X' & Z' \arrow[l, "f'"'] \arrow[r, "j'"] \arrow[d] & P' \arrow[dl, "g'"] \\ & Y' \arrow[from=1-1, "u'"'] & \end{tikzcd}$$

where j' is an open immersion, f' is projective and surjective, and g' is projective. Noting that finite type schemes over noetherian rings are automatically finitely presented, the diagram we seek is obtained from the above by base change along the ring map $A \to A'$. $\square$

4 First applications of limit formalism

For many of the following applications, we will reduce some statements about general schemes to statements about finite type $\mathbf{Z}$-schemes, and the analysis in this case will be aided by the following lemma.

Lemma 6. *Let R be a finite type $\mathbf{Z}$-algebra, and let $\mathfrak{p} \subset R$ be a prime ideal. Then $\mathfrak{p}$ is the intersection of the maximal ideals containing it. Moreover, if $\mathfrak{p}$ is maximal then $R/\mathfrak{p}$ is finite.*

Proof. This fact, too, is really a piece in a much more general context; the correct statement is that any finite type algebra over a Jacobson ring is itself Jacobson. (A Jacobson ring is a ring for which the conclusion of this Lemma holds, so this general result includes the Nullstellensatz as well.) Now suppose that the lemma is false, so there is some finite type $\mathbf{Z}$-algebra R, some nonmaximal prime ideal $\mathfrak{p} \subset R$, and some $f \in R \setminus \mathfrak{p}$ such that every maximal ideal of R containing $\mathfrak{p}$ also contains f. Let $\mathfrak{q}$ be a prime ideal of R maximal among those containing $\mathfrak{p}$ and not containing f, and replace R by $R/\mathfrak{q}$ so that we may assume that R is an integral domain which is not a field and yet R_f is a field.

With the above reductions made, let $\mathfrak{r}$ be the kernel of $\mathbf{Z} \to R$. If $\mathfrak{r} = (p)$ for some prime number p, then $\mathbf{F}_p \to R_f$ is a finite type map, so by the Nullstellensatz we see that the map is finite since R_f is a field. Thus $\mathbf{F}_p \to R$ is finite, so R must be a field, contradiction. Thus $\mathfrak{r} = (0)$. But now Chevalley's theorem shows that the image $\{(0)\}$ of $\operatorname{Spec} R_f$ in $\operatorname{Spec} \mathbf{Z}$ is constructible. This is not true, so we have reached another contradiction.

Note that by the same constructibility argument as in the previous paragraph, if $\mathfrak{p}$ is maximal then $\mathfrak{p} \cap \mathbf{Z} = (p)$ for some prime number p and thus the Nullstellensatz shows that $R/\mathfrak{p}$, being a finite extension of $\mathbf{F}_p$, is itself finite. $\square$

Recall that a morphism $f : X \to Y$ is **radicial** if for every field K, the induced map $X(K) \to Y(K)$ is injective. Equivalently, the map $\Delta_{X/Y}$:

$X \to X \times_Y X$ is surjective. In particular, every radicial map is separated and every monomorphism is radicial. In fact, f is radicial if and only if it is injective and for every $x \in X$, if $y = f(x)$ then the extension $k(y)/k(x)$ is purely inseparable.

We say that f is **étale** if it is flat, locally of finite presentation, and $\Delta_{X/Y}$ is an open immersion. Using Nakayama's lemma, it is easy to check that under the first two conditions, the final condition is equivalent to vanishing of $\Omega^1_{X/Y} = \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the quasicoherent ideal defining (the closure of) the image of $\Delta_{X/Y}$. (The notation $\Omega^1_{X/Y}$ is chosen because this is the sheaf of relative differentials of X over Y, but that will not be explained in this note.) Equivalently, f is flat, locally of finite presentation, and for every $y \in Y$, the fiber X_y over y is the disjoint union of schemes of the form $\operatorname{Spec} k'$, where k' is a finite separable extension of $k(y)$.

If $X = \operatorname{Spec} k'$ and $Y = \operatorname{Spec} k$ for fields k and k', so f corresponds to a field extension k'/k, the map f is radicial if and only if k'/k is purely inseparable, and f is étale if and only if k'/k is finite separable. So the following theorem can be regarded as a vast generalization of the fact that a finite separable, purely inseparable field extension is trivial.

Theorem 5. *If $f : X \to Y$ is étale and radicial, then it is an open immersion.*

Proof. Since f is étale, it is open by Theorem 4. So replacing Y by $f(X)$, we may assume that f is surjective and hence prove that it is an isomorphism. So f is surjective, radicial, and open, whence a universal homeomorphism. From this it follows that f is integral, so being locally of finite presentation it is in fact finite. By passing to an affine open cover we may assume that $Y = \operatorname{Spec} A$ is affine, so that $X = \operatorname{Spec} B$ for some finite flat A-algebra B of finite presentation. From this it follows that B is locally free over A, so by further localizing A we may assume that B is free over A, say of rank n. If $\mathfrak{m} \subset A$ is a maximal ideal, then $B/\mathfrak{m}B$ is a separable field extension of $A/\mathfrak{m}$ of degree n since f is étale, and since we have seen that f is monic it then follows that $A/\mathfrak{m} \to B/\mathfrak{m}B$ is an isomorphism, so indeed $n = 1$ and f is an isomorphism. $\square$

The following theorem and Theorem 9 contain as special cases the statement that any injective polynomial map from $\mathbf{C}^n$ to itself is surjective.

Theorem 6. *[EGA IV, 10.4.11] Let S be a scheme, let X be an S-scheme of finite type, and let $f : X \to X$ be an S-endomorphism of X. If f is radicial, then it is surjective.*

Proof. First note that the property of being radicial is preserved by arbitrary base change, and surjectivity of a map can be checked on fibers. So by replacing f by $f_s : X_s \to X_s$ for some $s \in S$, we may assume that $S = \operatorname{Spec} k$ for some field k. Writing $k = \varinjlim k_i$ for k_i the finitely generated $\mathbf{Z}$-subalgebras of k, we find that there is some i and some finite type k_i-scheme X_i such that $X_i \otimes_{k_i} k \cong X$, and moreover the map f can be descended to some map $f_i : X_i \to X_i$. By Lemma 4, after increasing i we may assume that f_i is radicial. As surjectivity is preserved by arbitrary base change, we may pass from f to f_i to assume that X is finite type over $\mathbf{Z}$.

Now note that Chevalley's theorem implies that $f(X) \subset X$ is locally constructible. If a locally constructible set in X contains all closed points, then it is equal to X. Indeed, to prove this we may assume $X = \operatorname{Spec} A$ for some finite type $\mathbf{Z}$-algebra A, so by part 3 of Lemma 2 there is some finite type $\mathbf{Z}$-algebra B and map $\operatorname{Spec} B \to \operatorname{Spec} A$ with image this constructible set. Then the corresponding map $\varphi : A \to B$ has kernel contained in each maximal ideal of A by hypothesis, so its kernel consists entirely of nilpotent elements by Lemma 5. So the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ is dense, hence contains a dense open, and we conclude by noetherian induction that the map is surjective. Hence indeed we need only prove that $f(X)$ contains all the closed points of X.

We have seen above that each closed point of X has a finite residue field, being a finite extension of some $\mathbf{F}_p$. If q is any prime power, then $X(\mathbf{F}_q)$ is finite. Indeed, any element of $X(\mathbf{F}_q)$ lies in some affine open of X, so because X is quasicompact it suffices to prove this when $X = \operatorname{Spec} A$. But then any element of $X(\mathbf{F}_q)$ is the same as a map $A \to \mathbf{F}_q$, and there are only finitely many such maps because A is finite type over $\mathbf{Z}$ and $\mathbf{F}_q$ is finite. Because f is radicial, the induced map $X(\mathbf{F}_q) \to X(\mathbf{F}_q)$ is injective and hence surjective by finiteness. So $f(X)$ contains all closed points of X and we have seen that this implies that f is surjective. $\square$

5 Fpqc descent for properties of morphisms

In the last few sections, we saw how to systematically reduce certain affine-local questions about schemes to stalk-local questions, i.e., to questions about local rings. Such questions can often further be reduced, as we have seen, to questions about noetherian local rings. In this section we will show that we can often further reduce to the case of *complete* noetherian local rings, which have several special features.

In fact, we will work in a much richer context which will show that many properties of morphisms of schemes can be checked after certain classes of base change. More specifically, a morphism $f : X \to Y$ is **fpqc** if it is faithfully flat and quasicompact. (In French, *fpqc* stands for *fidèlement plat et quasi-compact*.) If $\mathbf{P}$ is a property of morphisms of schemes, we will say that $\mathbf{P}$ *satisfies fpqc descent* if for any morphism $f : X \to Y$ and any fpqc morphism $g : Y' \to Y$, if $f' : X' \to Y'$ denotes the base change of f along g and f' is $\mathbf{P}$, then also f is $\mathbf{P}$. Almost all reasonable properties of scheme morphisms satisfy fpqc descent, and we will see many examples of this. One of the reasons to expect fpqc morphisms to be well-behaved comes from the fact that they are topological quotient maps. We will prove this after first proving the following two lemmas.

Lemma 7. *Let $f : X \to Y$ be a morphism of affine schemes such that $f(X)$ is closed under specialization, i.e., whenever $y_0 \in f(X)$ and $y \in Y$ is in the closure of y_0, the point y lies in $f(X)$. Then $f(X)$ is closed.*

Proof. Suppose $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, so that f is associated to some ring map $\varphi : A \to B$. Let $I = \ker \varphi \subset A$. Then the image of f is contained in $V(I)$ and we reduce easily to proving that f is surjective if φ is injective. By the hypothesis it suffices in this case to show that $f(X)$ contains every generic point of Y. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a minimal prime. Since $A_\mathfrak{p}$ is flat over A, it follows that the map $A_\mathfrak{p} \to B_\mathfrak{p} = B \otimes_A A_\mathfrak{p}$ is injective, and in particular $B_\mathfrak{p}$ is not the zero ring. From this it follows that $\mathfrak{p}$ lies in the image of f, so we are done. $\square$

Lemma 8. *Flat morphisms satisfy going-down, i.e., if $f : X \to Y$ is a flat morphism of schemes, $x_1 \in X$, $y_1 = f(x_1)$, and $y_0 \in Y$ is a generization of y_1, then there exists some $x_0 \in X$ such that $f(x_0) = y_0$.*

Proof. Replacing Y by $\operatorname{Spec} \mathcal{O}_{Y,y_1}$ and X by $\operatorname{Spec} \mathcal{O}_{X,x_1}$, we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ for A, B local rings and $A \to B$ flat. Moreover, replacing Y by $\overline{\{y_0\}}$, we may assume that Y is an integral scheme with generic point y_0. If K is the fraction field of A, then since $A \to K$ is injective and $A \to B$ is flat, the map $B \to B \otimes_A K$ is also injective and in particular $B \otimes_A K$ is not the zero ring. Since $f^{-1}(y_0) = \operatorname{Spec} B \otimes_A K$, any point $x_0 \in \operatorname{Spec} B \otimes_A K$ works to satisfy the conclusion of the lemma. $\square$

Theorem 7. *Fpqc morphisms are submersive, i.e., they are topological quotient maps.*

Proof. Let $f : X \to Y$ be an fpqc morphism of schemes, and let $U \subset Y$ be a subset such that $f^{-1}(U)$ is open in X. To prove the proposition we must show that U is open. Note first that if $\{V_i\}$ is an affine open cover of Y, then U is open in Y if and only if $Y \cap V_i$ is open in V_i for all i. So, replacing Y by one such V_i, we may assume that $Y = \operatorname{Spec} A$ is affine. In this case, X is quasicompact, so we may write $X = \bigcup_{j=1}^n U_j$ for affine open subsets $U_j \subset X$. if $g : \coprod_{j=1}^n U_j \to X$ is the natural morphism, then $f \circ g$ is fpqc since it is a faithfully flat morphism of affine schemes. Since $g^{-1}(f^{-1}(U))$ is open in $\coprod_{j=1}^n U_j$, we may replace X by $\coprod_{j=1}^n U_j$ to assume that $X = \operatorname{Spec} B$ is affine.

With the above reductions made, let Z be the complement of U in Y. To prove the result, it is enough to show that Z is closed. Note that $f^{-1}(Z)$ is the complement of $f^{-1}(U)$ in X, so $f^{-1}(Z)$ is closed. Since f is surjective, $U = f(f^{-1}(U))$, so because open sets are closed under generization it follows from Lemma 8 that U is closed under generization, and thus that Z is closed under specialization. But $f^{-1}(Z)$, being a closed subset of an affine scheme, is itself affine, and $Z = f(f^{-1}(Z))$ is closed under specialization, so by Lemma 7, it follows that Z is closed. This completes the proof. $\square$

To prove that certain properties satisfy fpqc descent, we need the following simple lemma.

Lemma 9. *Suppose $\boldsymbol{P}$ is a property of scheme morphisms such that*

1. *$\boldsymbol{P}$ is affine-local on the base,*
2. *for any faithfully flat morphism $V \to U$ of affine schemes and any morphism $f : X \to U$, if the morphism $f_V : X \times_U V \to V$ is $\boldsymbol{P}$ then f is $\boldsymbol{P}$.*

Then $\boldsymbol{P}$ satisfies fpqc descent.

Proof. Let $f : X \to Y$ be a morphism and let $g : Y' \to Y$ be fpqc. Suppose that the base change morphism $f' : X' \to Y'$ is $\mathbf{P}$. By 1, to prove that f is $\mathbf{P}$ we may assume that $Y = \operatorname{Spec} A$ is affine. Writing $Y' = \bigcup_{i=1}^n U_i'$ for affine open subschemes U_i' of X', again by 1 we see that the base change of f' along $\coprod_{i=1}^n U_i' \to Y'$ is $\mathbf{P}$, and so we may assume that $Y' = \operatorname{Spec} A'$ is affine. Thus 2 implies that f is $\mathbf{P}$. $\square$

Lemma 10. *The following properties* $\boldsymbol{P}$ *satisfy fpqc descent:*

1. *"f is affine"*
2. *"f is a closed immersion"*
3. *"f is finite"*
4. *"f is separated"*
5. *"f is locally of finite type"*
6. *"f is locally of finite presentation"*
7. *"f is quasicompact"*
8. *"f is quasiseparated"*
9. *"f is universally closed"*
10. *"f is finite type"*
11. *"f is finite presentation"*
12. *"f is proper"*

Proof. Note that all of these properties are Zariski local on the base, so it suffices to show that if $X \to \operatorname{Spec} A$ is a morphism of schemes, $A \to A'$ is faithfully flat, and $X_{A'} \to \operatorname{Spec} A'$ is $\mathbf{P}$, then $X \to \operatorname{Spec} A$ is $\mathbf{P}$.

For 1, recall that Serre's criterion for affineness states that a scheme X is affine if and only if $H^1(X, \mathcal{I}) = 0$ for every quasicoherent sheaf of ideals $\mathcal{I}$. If $\mathcal{I}$ is such a quasicoherent ideal sheaf, then we obtain a quasicoherent sheaf $\mathcal{I}_{A'}$ on $X_{A'}$, and because $X_{A'}$ is affine we have $H^1(X_{A'}, \mathcal{I}_{A'}) = 0$. By flat base change for cohomology it follows that $H^1(X_{A'}, \mathcal{I}_{A'}) = H^1(X, \mathcal{I}) \otimes_A A'$, so indeed $H^1(X, \mathcal{I}) = 0$ by faithful flatness of A' over A.

For 2, note that by 1 the map $X \to \operatorname{Spec} A$ is affine, so that $X = \operatorname{Spec} B$ for some A-algebra B. Suppose that $\operatorname{Spec} B \otimes_A A' \to \operatorname{Spec} A'$ is a closed immersion, so that $A' \to B \otimes_A A'$ is surjective. Since A' is faithfully flat over A, it follows immediately that $A \to B$ is surjective.

For 3, note again that by 1 we have $X = \operatorname{Spec} B$ for some A-algebra B. Suppose that $A' \to B \otimes_A A'$ is finite. Let $x_1, \ldots, x_n \in B \otimes_A A'$ be a finite set of generators for $B \otimes_A A'$ as an A'-module. Then we can write $x_i = \sum_{j=1}^{m_i} b_{ij} \otimes a'_{ij}$. We claim that $\{b_{ij}\}$ form a set of generators for B as an A-module. Indeed, if $m = \sum_{j=1}^{n} m_j$ then we have an A-module homomorphism $A^m \to B$ sending e_{ij} to b_{ij}. Since this map is surjective after tensoring by

A', it follows from faithful flatness that it is surjective to begin with, so we are done.

Point 4 follows directly from 2: notice that $f : X \to Y$ is separated if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed embedding. Since formation of the relative diagonal is compatible with base change, this follows directly from 2.

For 5, since the property of being locally finite type is Zariski local on the source, it suffices to consider the case that $X = \operatorname{Spec} B$ is affine. Then by hypothesis, $B \otimes_A A'$ is finite type over A', so there exist finitely many elements $b_1, \ldots, b_n \in B$ which generate $B \otimes_A A'$ as an A'-algebra. So the map $A[t_1, \ldots, t_n] \to B$ sending t_i to b_i is surjective after tensoring by A'. Since A' is faithfully flat over A, it follows that this map is surjective to begin with.

For 6, we reduce as in 5 to the case of an affine map $\operatorname{Spec} B \to \operatorname{Spec} A$, and we see that this map is finite type. So there is a surjection of A-algebras $A[t_1, \ldots, t_n] \to B$. Let K be the kernel of this map, so that by hypothesis $K \otimes_A A'$ is a finite type $A'[t_1, \ldots, t_n]$-module. As $K \otimes_A A' = K \otimes_{A[t_1,\ldots,t_n]} A'[t_1, \ldots, t_n]$ and $A'[t_1, \ldots, t_n]$ is faithfully flat over $A[t_1, \ldots, t_n]$, it follows that K is a finite type $A[t_1, \ldots, t_n]$-module, so we are done.

For 7, it suffices to show that X is quasicompact in the above setting. For $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$, note that $X \times_Y Y' \to X$ is fpqc, so in particular it is surjective and if $X \times_Y Y'$ is quasicompact then so is X.

Point 8 follows directly from 6, since f is quasiseparated if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is quasicompact.

For 9, suppose that $f' : X_{A'} \to \operatorname{Spec} A'$ is universally closed. By performing a base change to begin, it suffices to show that f is closed. For this, let $Z \subset X$ be a closed subset, so $Z_{A'}$ is a closed subset of $X_{A'}$ and thus $f'(X_{A'})$ is closed in $\operatorname{Spec} A'$. Since $\operatorname{Spec} A' \to \operatorname{Spec} A$ is a submersion by Theorem 7, it follows that $f(Z)$ is closed since $f^{-1}(Z)_{A'} = f'(Z_{A'})$

Point 10 follow directly from points 5 and 7. Point 11 follows from points 6, 7, and 8. Point 12 follows from points 4, 9, and 10. □

6 Further applications

Theorem 8. *Any proper finitely presented monomorphism of schemes is a closed immersion.*

Proof. This result remains true without the finite presentation hypothesis, but takes *much* more work to prove, and in particular the techniques of the proof go beyond this formalism. Suppose that the map $f : X \to Y$ is a proper finitely presented monomorphism. To show that it is a closed immersion we can pass to an affine open subset of Y to assume that $Y = \operatorname{Spec} A$. Writing A as the direct limit of its noetherian subrings allows us to further assume that Y is noetherian, from which it follows that X is also noetherian. Moreover, if $\mathfrak{p} \subset A$ is a prime ideal, then by Lemma 4 applied to $\varinjlim_{f \notin \mathfrak{p}} A_f = A_{\mathfrak{p}}$, since the property of being a closed immersion is affine local on the base we may assume that A is local. By Lemma 10 the property of being a closed immersion satisfies fpqc descent, so we may assume further that A is complete.

Now let $x \in X$ be the unique point in the fiber X_y over the closed point $y \in Y$. Then $\mathcal{O}_{X,x}$ is a finite A-module since A is complete: this can be proved from the fact that it is finite mod $\mathfrak{m}_A$ by quasi-finiteness and then using completeness of A to take successive approximations. Let $X' = \operatorname{Spec} \mathcal{O}_{X,x}$, so that the composite $X' \to X \to Y$ is finite and hence, $X \to Y$ being separated, $X' \to X$ is also finite, so X' is closed in X. Since g is finite type and X is locally noetherian, we may apply Lemma 3 to the inductive system of affine neighborhoods of x in X to see that there is an inverse map $U \to X'$ for some affine neighborhood of x in X, and by Lemma 4 we can arrange for it to be an isomorphism. So indeed X' is open in X, and thus we have $X = X' \coprod X''$ where X' is finite over Y and $f(X'')$ does not contain the closed point of Y.

But since f is proper, $f(X'')$ is closed in Y, so not containing the closed point means that it is empty. Hence $X = X'$ is finite over Y. So we have $X = \operatorname{Spec} B$ for some finite A-algebra B. If $\mathfrak{m}$ is the maximal ideal of A then the map $\operatorname{Spec} B/\mathfrak{m}B \to \operatorname{Spec} A/\mathfrak{m}$ is monic, so that the multiplication map $B/\mathfrak{m}B \otimes_{A/\mathfrak{m}} B/\mathfrak{m}B \to B/\mathfrak{m}B$ is bijective, and comparing $A/\mathfrak{m}$-dimensions then shows that the map $A/\mathfrak{m} \to B/\mathfrak{m}B$ is an isomorphism. So by finiteness of $A \to B$ and Nakayama's lemma, we see that $A \to B$ is surjective and thus f is a closed immersion. $\square$

Theorem 9. *[EGA IV, 17.9.6] Let S be a scheme, let X be an S-scheme of finite presentation, and let $f : X \to X$ be an S-endomorphism of X. If f is monic, then it is an isomorphism.*

Proof. We may assume that $S = \operatorname{Spec} A$ is affine, and since X is finite presentation over A it can be descended to lie over some finite type

$\mathbf{Z}$-subalgebra of S, as can the map f. Moreover, by Lemma 4 we can descend it in such a way that f is monic, and thus assume that X is finite type over $\mathbf{Z}$. By Theorem 6, using the fact that all monomorphisms are radicial, we see that f is surjective. So to show that f is an isomorphism, it suffices by Theorem 5 to show that it is étale. Let $x \in X$. Then the map $k(f(x)) \to k(x)$ is an isomorphism since f is monic, so that the map $\widehat{\mathcal{O}_{X,f(x)}} \to \widehat{\mathcal{O}_{X,x}}$ is an isomorphism on residue fields, and to show that f is étale at x it suffices to show that this map on completions is an isomorphism. Indeed, flatness can be checked on completions, so this would show that f is flat at x and then by part 3 of Lemma 1 and the fact that the residue field is trivial it would imply that f is étale at x by one of our equivalent conditions noted above.

Now the étale locus in X is open (since the flat locus and the locus where $\Omega^1_{X/Y}$ vanishes are both open), so in particular it is constructible and hence it suffices to prove that the map $\widehat{\mathcal{O}_{X,f(x)}} \to \widehat{\mathcal{O}_{X,x}}$ is an isomorphism for every closed point $x \in X$. To show this, it is enough to show that $\mathcal{O}_{X,f(x)}/\mathfrak{m}_{f(x)}^{n+1} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ is an isomorphism for all n and all closed points $x \in X$. Recall that every closed point of X has finite residue field and, for a given prime number p and integer $d \geq 1$, let $T_{p,d}$ denote the set of closed points of X with residue field of order dividing p^d. Let $\mathcal{I}$ be the quasicoherent ideal sheaf of X defined as the product of $\mathfrak{m}_x^{n+1}$ for all $x \in T_{p,d}$. Let X' be the closed subscheme of X defined by $\mathcal{I}$, so that X' is the disjoint union of $\operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ as x ranges over all the points of $T_{p,d}$, i.e., $X' = \operatorname{Spec} B$ for $B = \prod_{x \in T_{p,d}} \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$. The restricted map $f : X' \to X$ factors through a map $f' : X' \to X'$ since each map $\mathcal{O}_{X,f(x)} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ is local and hence factors through $\mathcal{O}_{X,f(x)}/\mathfrak{m}_{f(x)}^{n+1}$. Since f is monic and the inclusion $X' \to X$ is monic, it follows from purely categorical reasoning that f' is monic, and it will suffice to show that f' is an isomorphism.

Now because $\mathcal{O}_{X,x}$ is noetherian and has finite residue field for every $x \in T_{p,d}$ it follows that each $\mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ is finite and thus that $\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ is finite. Since X' is a finite disjoint union of the spectra of these rings, it follows that the map f' is finite, hence in particular proper. By Theorem 8, f' is therefore a closed immersion, so the maps $\mathcal{O}_{X,f(x)}/\mathfrak{m}_{f(x)}^{n+1} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$ are surjective. Thus f' corresponds to a surjective endomorphism $\varphi' : B \to B$. Since B is a *finite* ring, it follows that φ' is an isomorphism, completing the proof. $\square$